Comparative Analysis of the Solution to Fredholm Linear Integro Differential Equation by VIA and SEM

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Abstract

This study presents a comparative analysis of the Variational Iteration Method (VIM) and the Series Expansion Method (SEM) for solving Fredholm Integro-Differential Equations (FIDEs). Both methods were applied to an illustrative example, showcasing their effectiveness in addressing this class of equations. While SEM provided approximate solutions with acceptable accuracy, VIM demonstrated a distinct advantage by delivering exact solutions. The performance of the methods was evaluated through numerical experiments, with results presented using graphs and tables for clarity. SEM, though straightforward in its approach, exhibited slower convergence and reduced precision. On the other hand, VIM, employing correction functionals and Lagrange multipliers, consistently achieved high accuracy with minimal computational effort. The findings confirm that while both methods are effective, VIM is the more reliable and efficient approach for solving FIDEs. Its ability to produce exact solutions highlights its suitability for practical applications in mathematics and engineering.

Keywords: *Fredholm integro-differential equations, Variational Iteration method and Series Expansion Method*

1. Introduction

The background of the study of integro-differential equations is rooted in the broader study of differential equations and integral equations (Ejes, Nwaoburu & Davies 2024).

Some important problems in science and engineering can usually be reduced to a system of integral and integro-differential equations (Rabbani & Zarali, 2012). Pursuing analytical solutions to integro-differential equations represents a formidable yet crucial endeavor in mathematical analysis.

The Variational Iteration Method (VIM) and the Series Expansion Method (SEM) are two powerful approaches for solving Fredholm Linear Integro-Differential Equations (FIDEs). VIM utilizes iterative correction functionals to refine approximate solutions, offering a simple yet effective method that converges rapidly to the exact solution. In contrast, SEM expresses the unknown function as an infinite series, where each term is derived to improve the approximation of the solution.

Several authors have used have Used VIM and SIM in solving integro differential equation. Some have also made comparative analysis of different methods in solving integro differential equation. For instance, Ejes, Nwaoburu and Davies (2024), made a comparative analysis of the solution to Fredholm linear integro differential equations by ADM, MADM and Series Expansion Method, their findings indicate that while each method has its strengths, MADM demonstrates superior accuracy in most cases, making it a promising tool for handling complex integro-differential equations in numerical analysis. Asire and Najmudd (2023), presented a comparative analysis of the Adomian Decomposition Method (ADM), the Modified Adomian Decomposition Method (MADM), and the Variational Iteration Method (VIM). The primary objective of their research was to identify the most effective method between the three methods.

They said that the Adomian Decomposition Method (ADM), Modified Adomian Decomposition Method (MADM), and Variational Iteration Method (VIM) are efficient and effective methods for solving a wide range of problems. They said that the main advantage of these methods is that they do not require the variables to be discretized. Furthermore, these are unaffected by computation round off errors. Furthermore, they concluded that while the Adomian Decomposition Method (ADM) involves the computation of an Adomian polynomial, which demands time-intensive algebraic calculations, the Variational Iteration Method (VIM) requires only the evaluation of a Lagrangian multiplier. Additionally, VIM simplifies the computational process and provides solutions more quickly compared to both ADM and the Modified Adomian Decomposition Method (MADM).

Jackreece and Godspower (2017) made a comparison of Taylor Series and Variational Iteration method in solution of non- linear integro-differential equation. They observed that the Taylor Series methods seem to be more effective as the absolute errors are less than those from Variational iterative method.

Batiha, Noorani and Hashim (2006), use VIM to solve multi species Lotka-volterra equation. In comparisons with the Adomian decomposition and the fourth-order Runge–Kutta methods, they concluded that the variational iteration method is a powerful method for nonlinear equations.

2 Methodology

Let us consider the linear fredholmn integro differential equation

$$u^n(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \tag{2.1}$$

With $u^m(0) = \gamma_m, 0 \leq m \leq (n-1)$ $that\ is\ u(0) = \gamma_1, \frac{du(0)}{dx} = \gamma_2, \frac{d^2u(0)}{dx^2} = b_3, \frac{d^3u(0)}{dx^3} = \gamma_4 \dots\dots\dots\dots\dots.. \frac{d^{n-1}u(0)}{dx^{n-1}} = \gamma_{n-1}$

In this context $\gamma_1, \gamma_2, \gamma_3, \gamma_3, \dots\dots\dots\dots.. \gamma_{n-1}$ denotes real constants representing the initial condition of $u(x)$ and its derivatives at 0, while $u^n(x)$ which is equivalent to $\frac{d^nu}{dx^n}$ denotes the nth derivative of the unknown function $u(x)$ and $f(x)$ is a known function. These derivatives appear both inside and outside the integral sign. The integral function's kernel, denoted as $K(x,t)$, and the function $f(x)$ are specified as real-valued functions while u(t) represents a linear function of it.

The methods being discussed include the Variational Iteration Method and the Series Expansion Method, each of which has contributed to the progress in solving these types of equations. The following sections will provide a detailed explanation of each method.

2.1 Variational Iteration Method

This method is employed to solve a wide range of both linear and nonlinear equations, including Fredholm integro-differential equations and linear and nonlinear Volterra integro-differential equations, providing rapidly converging approximations to the exact solutions. The initial approximation can be chosen arbitrarily and may include unknowns that are determined using the initial conditions.

Let us consider the equation

$$Lu(x) + Nu(x) = f_{\sim}(x) \quad (2.2)$$

Where $L,\ N$ are linear and nonlinear operators respectively and $f_{\sim}(x)$ is a non-homogeneous term.

The correction functional for the above equation is given as:

$$u_{n+1}(x) = u_n(x) + \int_0^x \zeta(r)\,(Lu_n(r) + Nu_n(r) - f_{\sim}(r))dr \quad (2.3)$$

Where ζ is the Lagrange's multiplier which can be a constant or function and u_n is a restricted value which implies that it behaves as a constant, hence $\Delta u_n = 0$ where Δ is a variational derivative.

The following steps outline the application of the Variational Iteration Method:

First, the Lagrange multiplier $\zeta(r)$ is determined optimally. The result is then substituted into the correction functional, omitting the restriction. By taking the variation of the correction functional with respect to the independent variation u_n, we obtain

$$\frac{\Delta u_{n+1}}{\Delta u_n} = 1 + \frac{\Delta}{\Delta u_n}\left(\int_0^x \zeta(r)\,(Lu_n(r) + Nu_n(r) - f_{\sim}(r)dr\right) \quad (2.4)$$

Which is reduced for Fredholm linear integro differential equation to

$$\Delta u_{n+1} = \Delta u_n + \Delta\left(\int_0^x \zeta(r)\,(Lu_n(r)dr\right) \quad (2.5)$$

Appling integration by part to get the value of the Lagrange multiplier $\zeta(r)$. We get

First order

$$\int_0^x \zeta(r)\, u'_n(r)dr = \zeta(r)u_n(r) - \int_0^x \zeta'(r)\, u_n(r)dr \quad (2.6)$$

Second order

$$\int_0^x \zeta(r)\, u''_n(r)dr = \zeta(r)u'_n(r) - \zeta'(r)u_n(r) \int_0^x \zeta''(r)\, u_n(r)dr \quad (2.7)$$

Third order

$$\int_0^x \zeta(r)\, u'''_n(r)dr = \zeta(r)u''_n(r) - \zeta'(r)u'_n(r) + \zeta''(r)u_n \int_0^x \zeta'''(r)\, u_n(r)dr \quad (2.5)$$

Forth order

$$\int_0^x \zeta(r)\, u^{iv}_n(r)dr = \zeta(r)u'''_n(r) - \zeta'(r)u''_n(r) + \zeta''(r)u'_n - \zeta'''(r)u_n(r) + \int_0^x \zeta'''(r)\, u_n(r)dr \quad (2.6)$$

And so on. The identities are all gotten via integration by part.

The variational principle requires that the correction functional satisfies

$$\Delta u_{n+1} = u_{n+1} - u_n = 0. \quad (2.7)$$

This implies that

for first order

$\zeta(r) = -1$

for second order

$\zeta(r) = r - x$

for third order

$\zeta(r) = -\frac{(r-x)^2}{2}$

For nth order,

$$\zeta(r) = \frac{(-1)^n (r-x)^{n-1}}{(n-1)!}. \quad (2.8)$$

With the Lagrange multiplier determined, next we obtain the successive approximation $u_{n+1}, n \geq 1$, of the solution $u(x)$, which will be gotten using selective functional $u_o(x)$.

$u_o(x)$ is selected the initial conditions

$u_o(x) = u(0)$ for order one

$u_o(x) = u(0) + xu'(0)$ for order two

$u_o(x) = u(0) + xu'(0) + \frac{x^2}{2!}u''(0)$ for order three

$u_o(x) = u(0) + xu'(0) + \frac{x^2}{2!}u''(0) + \frac{x^3}{3!}u'''(0)$ for order four

$$u_o(x) = u(0) + xu'(0) + \frac{x^2}{2!}u''(0) + \frac{x^3}{3!}u'''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}u^{n-1}(0) \text{ for order n} \quad (2.9)$$

Hence $u(x) = \lim_{n\to\infty} u_n(x)$ (2.10)

2.2 Series Expansion Method

The Taylor series method represents the solution as a power series expansion. It involves expanding the unknown function and the kernel function in Taylor series about a given point and substituting these expansions into the integral equation. By equating coefficients of like powers of x, one can obtain a sequence of equations for the coefficients of the series expansion, which can then be solved to approximate the solution (Ejes, Nwaoburu & Davies 2024).

The Series Solution Method is fundamentally based on the use of Taylor series expansions for analytical functions. It is crucial to note that the applicability of Taylor series requires the existence of derivatives of all orders, necessitating their computation. Additionally, a Taylor series centered at any point b within its domain converges to f(x) within a neighborhood around b

$$u(x) = \sum_{n=0}^{\infty} \frac{u^n(a)}{n!}(x-b)^n \quad (2.11)$$

When x = 0, equation (3.29) is reduced to

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad (2.12)$$

Or $u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (2.13)

It is normal to integrate both sides of equation (2.1). Suppose L^{-1} is an n – fold integration operator

$$L^{-1}(u^n(x)) = L^{-1}(f(x)) + L^{-1}(\lambda \int_a^b K(x,t)u(t))dt) \quad (2.14)$$

$$u(x) = \gamma_0 + \gamma_1 x + \frac{1}{2!}\gamma_2 x^2 + \frac{1}{3!}\gamma_3 x^3 + \cdots + \frac{1}{(n-1)!}\gamma_{n-1} x^{x-1} + L^{-1}(f(x)) + L^{-1}(\lambda(\int_a^b k(x,t)u(t)dt)) \quad (2.15)$$

Equation (2.15) can be expressed as

$$u(x) = \sum_{l=0}^{n-1} \frac{1}{l!}\gamma_l x^l + L^{-1}(f(x)) + L^{-1}(\lambda \int_a^b K(x,t)u(t))dt) \quad (2.16)$$

Without loss of generality, if

$$K(x,t) = q(x)w(t) \tag{2.17}$$

Equation (2.17) implies that the kernel is separable,

Equation 2.4 can be expressed as

$$u(x) = \sum_{l=0}^{n-1} \frac{1}{l!}\gamma_l x^l + h(x) + L^{-1}q(x)(\lambda \int_a^b w(t)u(t))dt) \tag{2.18}$$

$\sum_{l=0}^{n-1} \frac{1}{l!}\gamma_l x^l$ is gotten from the n-fold integrator operation

From (2.18)

$$u(x) = \sum_{l=0}^{n-1} \frac{1}{l!}\gamma_l x^l + h(x) + L^{-1}q(x)(\lambda \int_a^b w(t)u(t))dt)$$

Substituting equation (2.12) into equation (2.18), we get

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{l=0}^{n-1} \frac{1}{l!}\gamma_l x^l + h(x) + L^{-1}q(x)(\lambda \int_a^b w(t) \sum_{n=0}^{\infty} a_n x^n)dt) \tag{2.19}$$

If $h(x)$ and $L^{-1}q(x)$ comprises elementary functions like exponential functions, trigonometric functions, etc., we should employ Taylor expansions for the functions contributing to the function. Now, equating coefficients of like powers of x on both sides, we obtain a system of equations for the coefficients a_n. Solving this system will give us the coefficients and hence the Taylor series solution for equation.

2.3 Solved Example

Example 2.3.1: Consider the linear Fredholm integro-differential equation: $u'(x) = e^x - x + xe^x + \int_0^1 xu(t)dt$, with the initial condition $u(0) = 0$, and the exact solution is

$u(x) = xe^x$.(Asiya & Najmuddin, 2023).

Variational Iteration Method

From equation 2.3, the correction functional which is

$u_{n+1}(x) = u_n(x) + \int_0^x \zeta(r)\,(Lu_n(r) + Nu_n(r) - f_{\sim}(r))dr$ can be expressed as

$$u_{n+1}(x) = u_n(x) + \int_0^x \zeta(r)\,(u'_n(r) - e^r + r - re^r - \int_0^1 ru_n(t)dt)dr$$

Let $u_0(x) = u(0) = 0$

When $n = 0$

$$u_1(x) = u_0(x) + \int_0^x \zeta(r)\,(u'_0(r) - e^r + r - re^r - \int_0^1 ru_0(t)dt)dr$$

$\zeta(r) = -1$ since $L = \frac{d}{dx}$

Putting $u_0(x) = 0$ and $\zeta(r) = -1$ into the correctional functional

$$u_1(x) = -\left(\int_0^x (0 - e^r + r - re^r - \int_0^1 r(0)dt)\right)dr$$

$$u_1(x) = -\left(\int_0^x (- e^r + r - re^r\right)dr = \left(\int_0^x (e^r - r + re^r\right)dr$$

$$u_1(x) = \Big|_0^x \left[re^r - \frac{r^2}{2}\right] = xe^x - \frac{x^2}{2}$$

$$u_1(x) = xe^x - \frac{x^2}{2}$$

For $u_2(x)$

$$u_2(x) = u_1(x) - \int_0^x (u'_1(r) - e^r + r - re^r - \int_0^1 ru_1(t)dt)dr$$

$$u'_1(r) = e^r - r + re^r$$

$$\int_0^1 ru_1(t)dt = \frac{5}{6}r$$

So, $u_2(x) = xe^x - \frac{x^2}{12}$

$$u_3(x) = u_2(x) - \int_0^x (u'_2(r) - e^r + r - re^r - \int_0^1 ru_2(t)dt)dr$$

$$u'_2(r) = e^r - \frac{r}{6} + re^r$$

$$\int_0^1 ru_2(t)dt = \frac{35}{36}r$$

So, $u_3(x) = xe^x - \frac{x^2}{72}$

Similarly, $u_4(x) = xe^x - \frac{x^2}{432}$

$$u_5(x) = xe^x - \frac{x^2}{2592}$$

Observing the pattern, $u_n(x) = xe^x - \frac{x^2}{2(6)^{n-1}}$

$u(x) = \lim_{n\to\infty} u_n(x) = xe^x$ as $-\frac{x^2}{2(6)^{n-1}} \to 0$

Hence, $u(x) = xe^x$ which is the exact solution

Series Expansion Method

Using inverse operator $L^{-1} = \int(.)dx$ on example 2.3.1

We get $L^{-1}(u'(x)) = L^{-1}\,(e^x) - L^{-1}(x) + L^{-1}(xe^x) + L^{-1}(x\int_0^1 u(t)dt)$

$$\int u'(x)dx = \int e^x dx + \int xe^x dx + \int_0^1 u(t)dt)$$

$$u(x) = e^x - \frac{x^2}{2} + xe^x - e^x + \frac{x^2}{2}\int_0^1 u(t)dt + c$$

Where c is the constant of integration. Using the initial condition $u(0) = 0$

$$c = 0$$

We get

$u(x) = xe^x - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 u(t)dt.$

Let $u(x) = \sum_{n=0}^{\infty} a_n x^n$

$u(x) = xe^x - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 u(t)dt$. Can be written as

$\sum_{n=0}^{\infty} a_n x^n = xe^x - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 \sum_{n=0}^{\infty} a_n t^n\, dt$

Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

On substitution, $\sum_{n=0}^{\infty} a_n x^n = x\sum_{n=0}^{\infty} \frac{x^n}{n!} - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 \sum_{n=0}^{\infty} a_n t^n \, dt$

$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 \sum_{n=0}^{\infty} a_n t^n \, dt$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} - \frac{x^2}{2} + \frac{x^2}{2}\left[\sum_{n=0}^{\infty} a_n \frac{t^{n+1}}{n+1}\right|_0^1$$

$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} - \frac{x^2}{2} + \frac{x^2}{2}\sum_{n=0}^{\infty} \frac{a_n}{n+1}$ (ii)

Equation (ii) can be written as

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \left(x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} \dots\right) - \frac{x^2}{2} + \frac{x^2}{2}\left(a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots\right)$$

Comparing co-efficient

$$a_0 = 0$$

$$a_1 = 1$$

$$a_3 = \frac{1}{2!} = \frac{1}{2}$$

$$a_4 = \frac{1}{3!} = \frac{1}{6}$$

$$a_5 = \frac{1}{4!} = \frac{1}{24}$$

Let's calculate an approximate value for a_2

$$a_2 \approx 1 - \frac{1}{2} + \frac{1}{2}\left(a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} + \frac{a_4}{5} \dots\right)$$

$$a_2 \approx 1 - \frac{1}{2} + \frac{1}{2}\left(0 + \frac{1}{2} + \frac{a_2}{3} + \frac{\left(\frac{1}{2}\right)}{4} + \frac{\left(\frac{1}{6}\right)}{5} + \frac{\left(\frac{1}{24}\right)}{6} \dots\right)$$

$$a_2 \approx 1 - \frac{1}{2} + \frac{1}{2}\left(0 + \frac{1}{2} + \frac{a_2}{3} + \frac{1}{8} + \frac{1}{30} + \frac{1}{144} \dots\right)$$

$$a_2 - \frac{a_2}{6} \approx 1 - \frac{1}{2} + \frac{1}{2}\left(0 + \frac{1}{2} + \frac{1}{8} + \frac{1}{30} + \frac{1}{144} \dots\right)$$

$$a_2 - \frac{a_2}{3} \approx 0.8326$$

On evaluation, $a_2 \approx 0.999$

Hence the series becomes $u(x) \approx x + 0.999\frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} \dots$

3 Result

Table 3.1: Exact and Approximate Solution by ADM, MADM and SEM For Example 1with step size 0.01

x	EXACT	VIM	SEM	|Ex-VIM|	|Ex-SEM|
0.01	0.010100502	0.010100502	0.010050452	0	5.005E-05
0.02	0.020404027	0.020404027	0.020203827	0	0.000200199
0.03	0.030913636	0.030913636	0.03046319	0	0.000450446
0.04	0.041632431	0.041632431	0.040831647	0	0.000800784
0.05	0.052563555	0.052563555	0.051312354	0	0.001251201
0.06	0.063710193	0.063710193	0.061908516	0	0.001801677
0.07	0.075075573	0.075075573	0.072623388	0	0.002452185
0.08	0.086662965	0.086662965	0.083460282	0	0.003202683
0.09	0.098475686	0.098475686	0.094422566	0	0.00405312
0.1	0.110517092	0.110517092	0.105513667	0	0.005003425
0.11	0.122790588	0.122790588	0.116737073	0	0.006053515
0.12	0.135299622	0.135299622	0.128096337	0	0.007203286
0.13	0.14804769	0.14804769	0.139595078	0	0.008452612
0.14	0.161038332	0.161038332	0.151236983	0	0.009801349
0.15	0.174275136	0.174275136	0.163025813	0	0.011249324
0.16	0.187761739	0.187761739	0.174965398	0	0.012796341
0.17	0.201501825	0.201501825	0.187059649	0	0.014442176
0.18	0.215499125	0.215499125	0.199312551	0	0.016186574
0.19	0.229757424	0.229757424	0.211728174	0	0.01802925
0.2	0.244280552	0.244280552	0.224310667	0	0.019969885

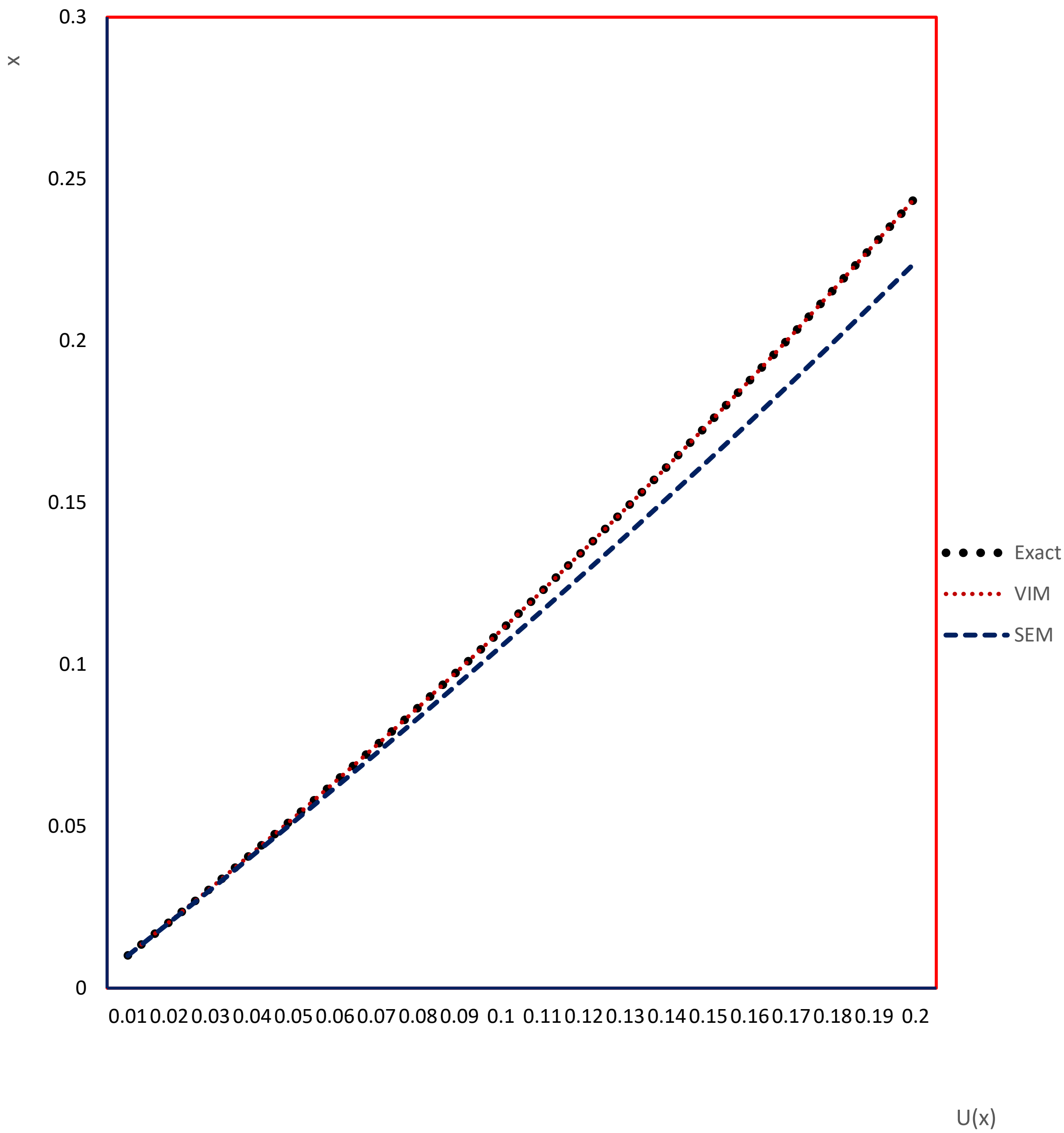

Figure 3.1. Exact and Approximate Solution by VIM and SEM with step size 0.01

Table 3.2: Exact and Approximate Solution by ADM, MADM and SEM For Example 1with step size 0.05

x	EXACT	VIM	SEM	|Ex-VIM|	|Ex-SEM|
0.05	0.052563555	0.052563555	0.051312354	0	0.001251201
0.1	0.110517092	0.110517092	0.105513667	0	0.005003425
0.15	0.174275136	0.174275136	0.163025813	0	0.011249324
0.2	0.244280552	0.244280552	0.224310667	0	0.019969885
0.25	0.321006354	0.321006354	0.289877604	0	0.03112875
0.3	0.404957642	0.404957642	0.360291	0	0.044666642
0.35	0.496673642	0.496673642	0.436177729	0	0.060495913
0.4	0.596729879	0.596729879	0.518234667	0	0.078495212
0.45	0.705740483	0.705740483	0.607236188	0	0.098504296
0.5	0.824360635	0.824360635	0.704041667	0	0.120318969
0.55	0.95328916	0.95328916	0.809602979	0	0.143686181
0.6	1.09327128	1.09327128	0.924972	0	0.16829928
0.65	1.245101539	1.245101539	1.051308104	0	0.193793435
0.7	1.409626895	1.409626895	1.189885667	0	0.219741229
0.75	1.587750012	1.587750012	1.342101563	0	0.24564845
0.8	1.780432743	1.780432743	1.509482667	0	0.270950076
0.85	1.988699824	1.988699824	1.693693354	0	0.29500647
0.9	2.2136428	2.2136428	1.896543	0	0.3170998
0.95	2.456424176	2.456424176	2.119993479	0	0.336430697
1	2.718281828	2.718281828	2.366166667	0	0.352115162

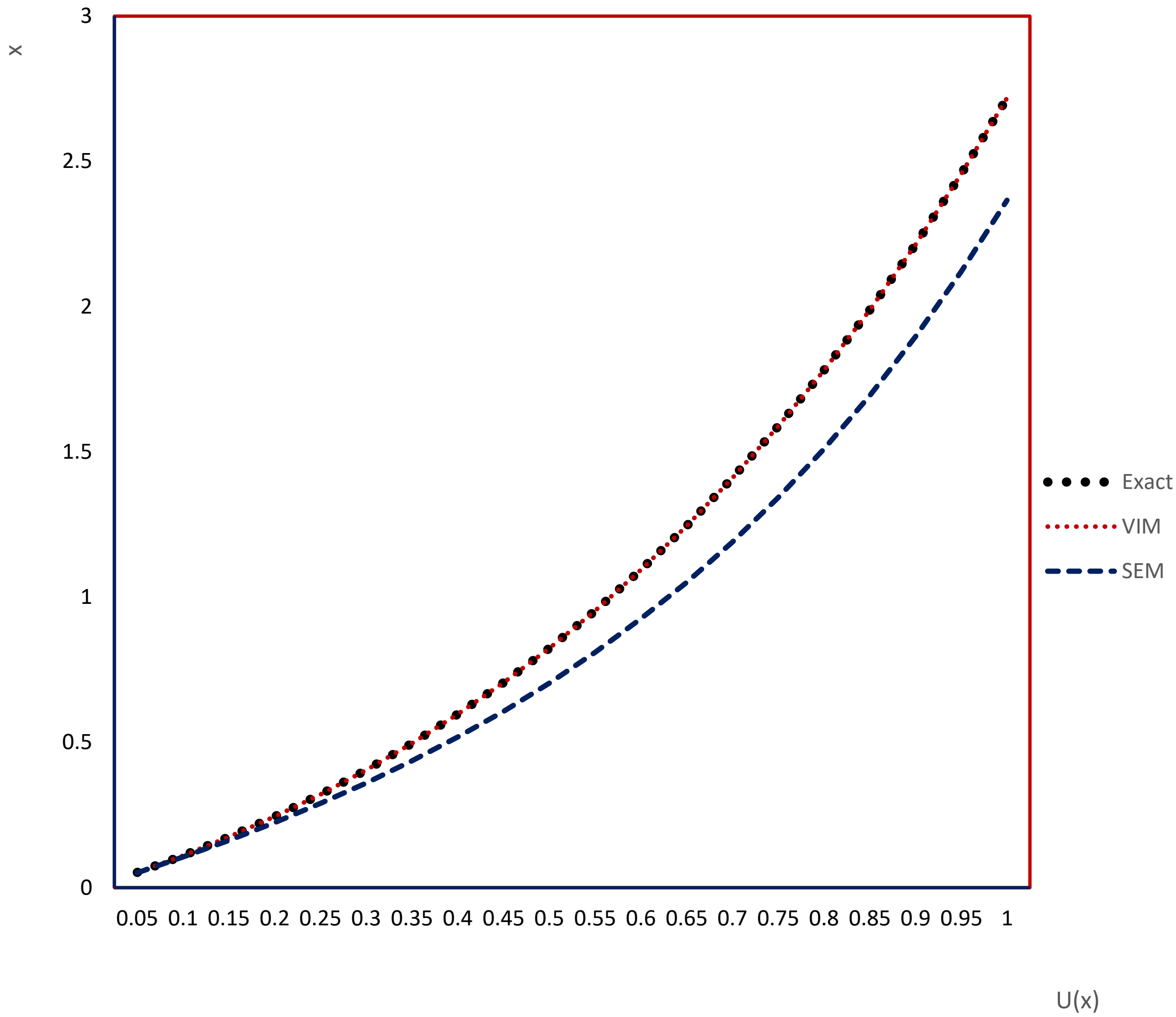

Figure 3.2. Exact and Approximate Solution by VIM and SEM with step size 0.05

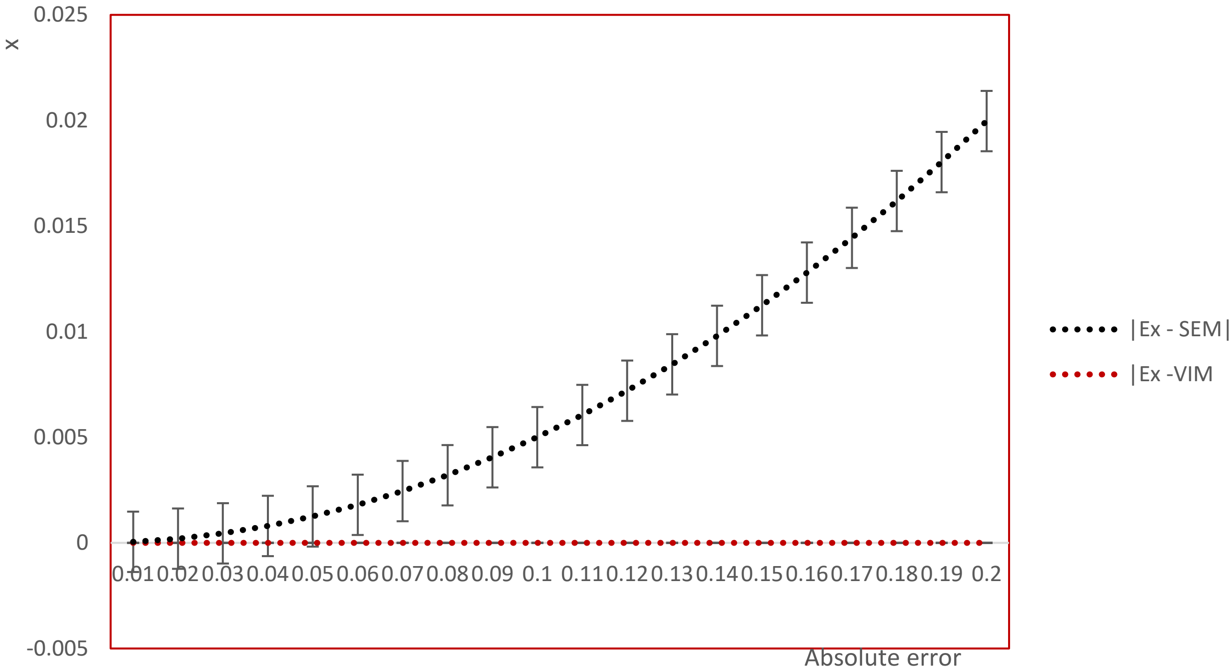

Figure 3.3: Absolute error with step size 0.01

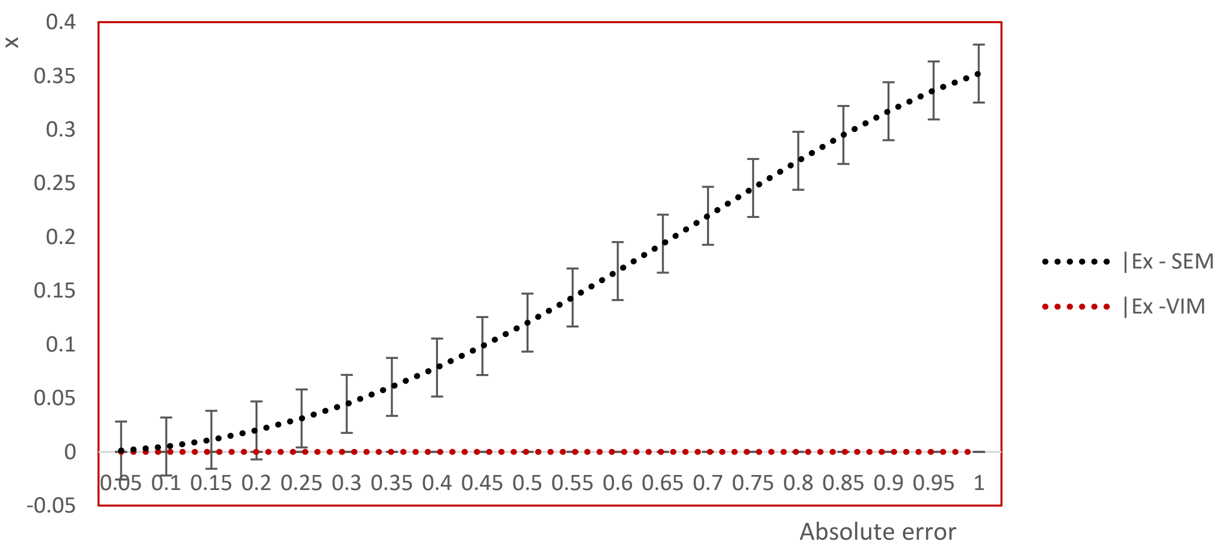

Figure 3.4. Absolute error with step size 0.05

Table 3.3. Root Mean Square Error with step size 0.01

Root Mean Square Error (RMSE)	
MADM	0
SEM	0.00951404

Table 3.4. Root Mean Square Error with step size 0.05

Root Mean Square Error (RMSE)	
VIM	0
SEM	0.240693898

4 Discussion

In this comparative analysis, we evaluated the solutions to Fredholm linear integro-differential equations using two different methods: the Variational Iteration Method (VIM) and the Series Expansion Method (SEM). The effectiveness of each method was assessed based on numerical accuracy, convergence rate, computational efficiency, and ease of implementation. Notably, the VIM is a highly effective approach, providing the exact solution, which makes it a superior method for solving these equations.

The VIM is a robust analytical method that constructs correction functionals using Lagrange multipliers, iteratively refining approximations to yield the exact solution. This method efficiently handles Fredholm integro-differential equations by iteratively improving the approximation without requiring complex transformations or restrictive assumptions. The accuracy and convergence speed of VIM surpass those of SEM, making it an ideal approach for solving such problems.

The Series Expansion Method involves expressing the solution as a series and determining the coefficients through various techniques, such as power series or Fourier series. However, compared to VIM, the convergence rate of SEM is slower. SEM also requires careful selection of the series type and precise computation of coefficients, making its implementation more challenging. While SEM can provide approximate solutions, its effectiveness diminishes for problems where high accuracy is required.

When comparing these methods, the results demonstrate the strengths and effectiveness of both techniques. VIM, however, consistently provides more precise approximations and often yields exact solutions. When applied to Fredholm linear integro-differential equations with separable kernels, the outcomes obtained through VIM and SEM are comparable in structure, but VIM's solutions exhibit superior precision and computational efficiency. Moreover, VIM achieves rapid convergence with fewer computational steps, whereas SEM requires more iterations to attain similar accuracy.

Tables 3.1–3.4 illustrate the comparative results, absolute errors, and root mean square errors (RMSE) of VIM and SEM in relation to exact solutions. The error analyses confirm that VIM outperforms SEM in terms of accuracy and convergence speed. Additionally, statistical assessments highlight that VIM achieves higher precision more rapidly than SEM. Furthermore, when the step size was increased from 0.01 to 0.05, SEM exhibited a tendency to deviate more from the exact solution and it still lagged behind VIM in terms of accuracy and efficiency. The RMSE table also indicates that VIM has zero error compared to SEM

The visual representations in Figures 3.1 to 3.4 complement these findings, providing a graphical overview of the analysis and reinforcing the superiority of VIM over SEM in solving Fredholm linear integro-differential equations.

5. Conclusion

The Variational Iteration Method (VIM) has proven to be a superior approach for solving Fredholm linear integro-differential equations, consistently delivering exact solutions with high accuracy and efficiency. Its rapid convergence and minimal computational effort make it an optimal choice compared to the Series Expansion Method (SEM), which, while useful, exhibits slower convergence and greater complexity in implementation. The comparative analysis demonstrated that VIM not only provides more precise approximations but also requires fewer computational steps, reinforcing its effectiveness. The numerical results, graphical representations, and statistical evaluations confirm VIM’s dominance in solving these equations. Consequently, for researchers and practitioners seeking an efficient and accurate method, VIM stands as the preferred approach.

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